VECTOR CALCULUS ON DISCRETE RIEMANNIAN MANIFOLDS

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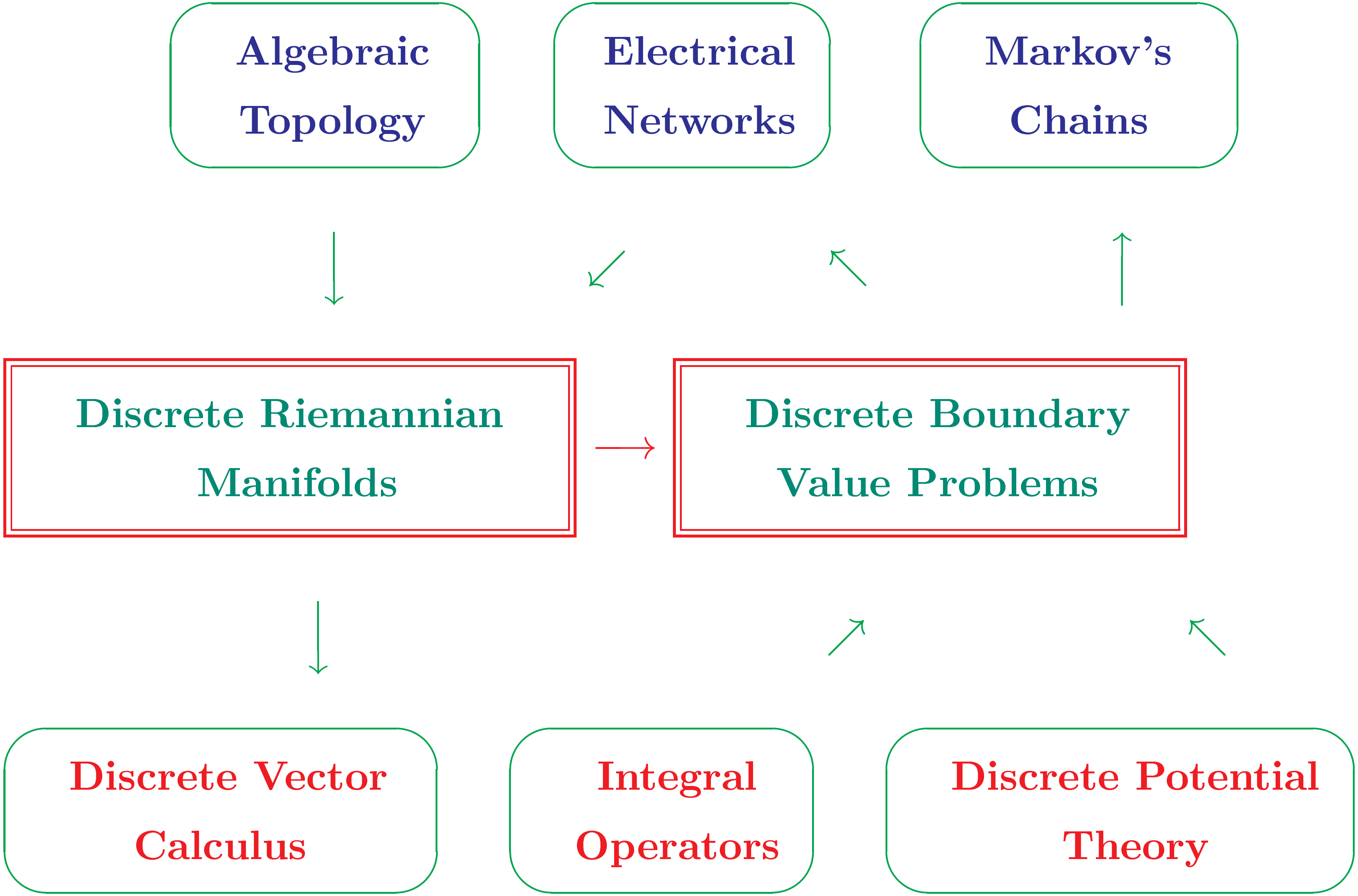
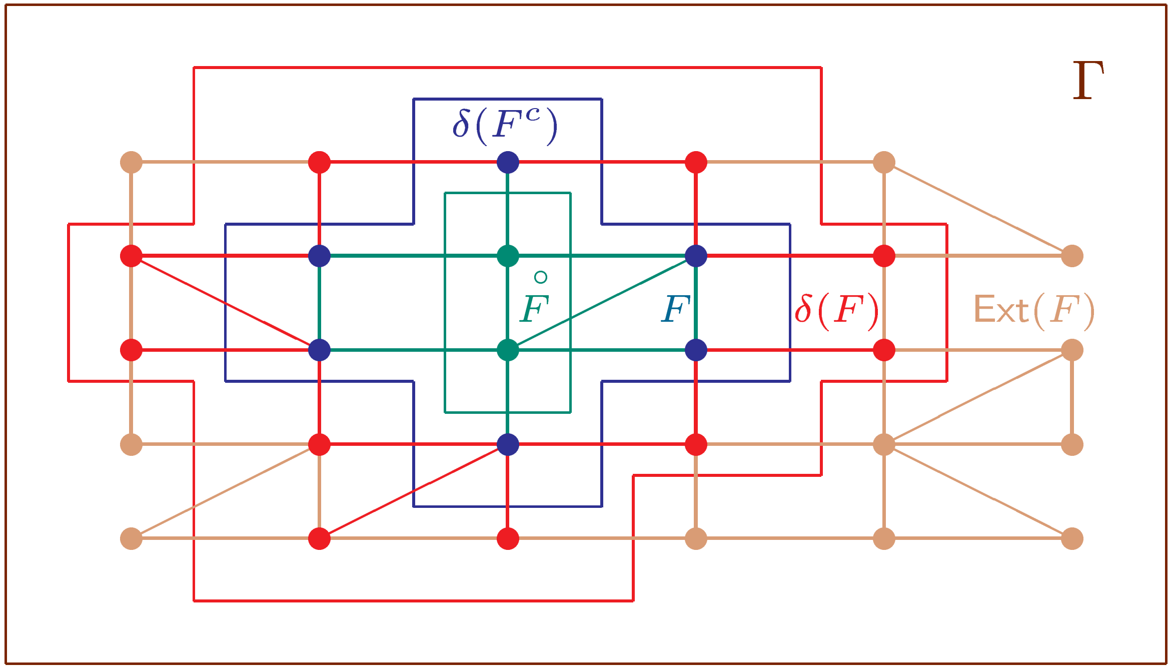

Algebraic Topology
Electrical Networks
Markov's Chains
Discrete Riemannian Manifolds
Discrete Boundary Value Problems
Discrete Vector Calculus
Integral Operators
Discrete Potential Theory

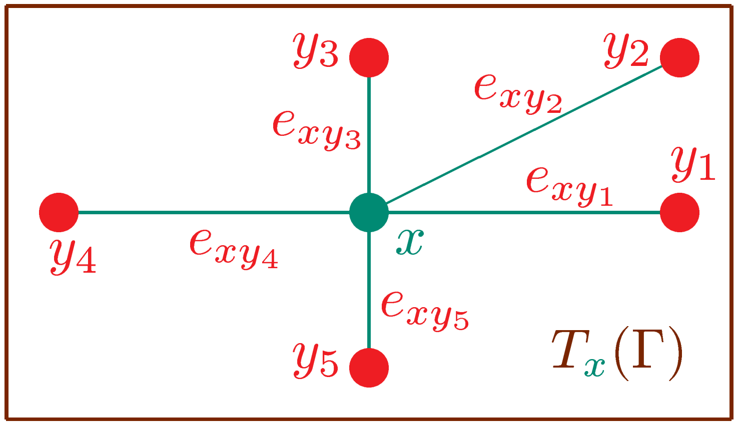
Discrete Manifolds

- $\Gamma = (V, E)$: Finite Graph
- $E_x = \bigcup_{y \sim x} e_{xy}$, $\quad |E_x| = k(x) \equiv$ Degree of x
- $F \subset V$



- $\delta(F)$: Vertex boundary
- $\delta(F^c)$: Interior vertex boundary
- $F = \mathring{F} \cup \delta(F^c)$
- $\bar{F} = F \cup \delta(F)$: Closure of F

Vector fields on discrete manifolds



▷ Tangent Space at x : $T_x(\Gamma)$

Base of $T_x(\Gamma)$: $\left\{e_{xy_j}\right\}_{j=1}^{k(x)}$

▷ Vector field in Γ: $\mathsf{f}(x) \in T_x(\Gamma)$

$$\mathsf{f}(x) = \sum_{j=1}^{k(x)} f(x, y_j)\, e_{xy_j} = \sum_{y\sim x} f(x,y)\, e_{xy}$$

⋆ Basic decomposition: $\mathcal{X}(\Gamma) = \mathcal{X}^s(\Gamma) \oplus \mathcal{X}^a(\Gamma)$

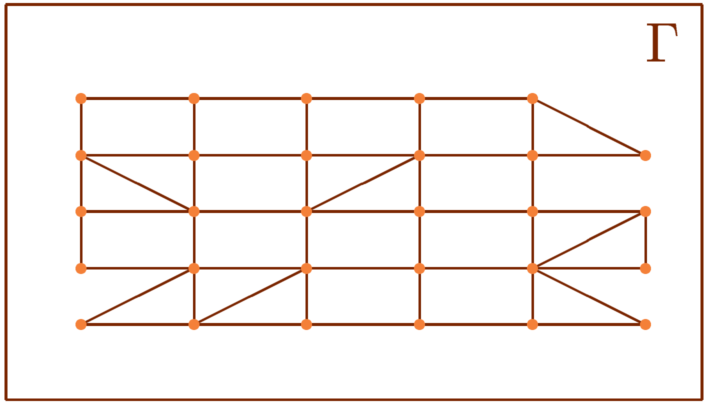
$$\mathsf{f}^s(x) = \tfrac{1}{2}\sum_{y\sim x}\Big(f(x,y) + f(y,x)\Big)\, e_{xy}, \quad \mathsf{f}^a(x) = \tfrac{1}{2}\sum_{y\sim x}\Big(f(x,y) - f(y,x)\Big)\, e_{xy}$$

▷ Field of bilinear applications on Γ: $\mathsf{B}(x) \in \mathcal{B}_x(\Gamma)$

▷ Field of matrices on Γ: $\mathsf{M}(x) \in \mathcal{M}_{k(x)}(\mathbb{R}) \Longrightarrow m(x,y,z), \quad y\sim x, \; z \sim x$

▷ Order of an operator: $\mathsf{F}\colon \mathcal{V} \longrightarrow \mathcal{X}(\Gamma), \quad \mathsf{H}\colon \mathcal{V} \longrightarrow \mathcal{C}(V), \quad \mathcal{V} \subset \mathcal{X}(\Gamma), \mathcal{C}(V)$

Discrete Riemannian manifolds



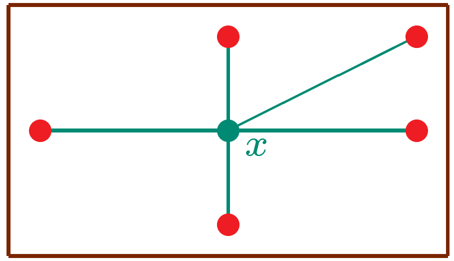
- ν, μ dense and positive measures on V
- B field of symmetric and positive definite bilinear applications
- Riemannian structure: $(\mathsf{B}, \mu) \Rightarrow \mathsf{M}$, $\mathsf{A} = \mathsf{M}^{-1}$ sym. p. d. fields of matrices
- Inner product on the vertex function space:

$$\int_V u\, v\, d\nu = \sum_{x \in V} u(x)\, v(x)\, \nu(x), \qquad u, v \in \mathcal{C}(V)$$

- Inner product on the vector field space:

$$\frac{1}{2} \int_V \mathsf{B}(\mathsf{f}, \mathsf{g})\, d\mu = \frac{1}{2} \sum_{x \in V} \langle \mathsf{M}(x) \mathsf{f}(x), \mathsf{g}(x) \rangle\, \mu(x), \qquad \mathsf{f}, \mathsf{g} \in \mathcal{X}(\Gamma)$$

Difference Operators



▷ Derivative of $u \in \mathcal{C}(V)$: $\boxed{\mathsf{d}u = -2u^a}$ $\Longrightarrow$ $\mathsf{d}u(x) = \sum_{y\sim x} \big(u(y) - u(x)\big)\, e_{xy}$

▷ Gradient of $u \in \mathcal{C}(V)$: $\boxed{\nabla u = \mathsf{A}\,\mathsf{d}u}$ $\Longrightarrow$ $\nabla u(x) = \sum_{y\sim x} \Big[\sum_{z\sim x} a(x,y,z)\,\big(u(z) - u(x)\big)\Big]\, e_{xy}$

▷ Divergence of $\mathsf{f} \in \mathcal{X}(\Gamma)$: $\boxed{\mathsf{div} = -\nabla^*}$ $\Longrightarrow$ $\int_V u\, \mathsf{div}\, \mathsf{f}\, d\nu = -\frac{1}{2}\int_V \mathsf{B}(\mathsf{f}, \nabla u)\, d\mu$

$$\mathsf{div}\ \mathsf{f}(x) = \frac{1}{2\nu(x)} \sum_{e\in E_{xy}} \big(\mu(x) f(x,y) - \mu(y) f(y,x)\big) = \frac{1}{\nu(x)} \sum_{y\sim x} (\mu f)^a(x,y)$$

▷ Curl of $\mathsf{f} \in \mathcal{X}(\Gamma)$: $\boxed{\mathsf{curl}\, \mathsf{f} = \frac{2}{\mu}(\mathsf{M}\mathsf{f})^s}$ $\Longrightarrow$ $\mathsf{curl}^* = \mathsf{curl}$, $\mathsf{div} \circ \mathsf{curl} = 0$, $\mathsf{curl} \circ \nabla = 0$

Laplace-Beltrami Operator

▷ Laplacian of $u \in \mathcal{C}(V)$: $\boxed{\Delta u = \mathsf{div}\,(\nabla u) = \mathsf{div}\,(\mathsf{A}\,\mathsf{d}u)}$

$$\Delta u(x) = \frac{1}{\nu(x)} \sum_{y \in V} c(x,y)\big(u(y) - u(x)\big)$$

$$c(x,y) = \frac{1}{2} \sum_{z \in V} \Big[a(x,z,y)\,\mu(x) + a(y,x,z)\,\mu(y) - a(z,x,y)\,\mu(z)\Big], \qquad x \neq y$$

⋆ $-\Delta$ is a second order self-adjoint and positive semidefinite operator

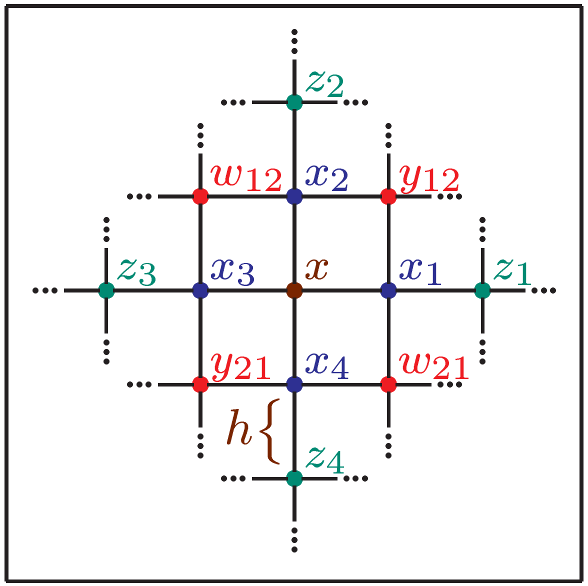
$$\Delta u = 0 \quad \text{iff} \quad u = \text{cte on each connected component of } \; \Gamma$$

Difference Schemes

▷ Differential operator: $\mathcal{L}u = -\mathrm{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u$

▷ The Stencil: $S_h(x) = \{x\} \bigcup \{x_j, z_j\}_{j=1}^{2n} \bigcup \{y_{ij}, w_{ij}\}_{1\le i<j\le n}$:

$$\begin{aligned}
x_j &= x + h\mathsf{e}_j, & x_{n+j} &= x - h\mathsf{e}_j, & j &= 1,\dots,n,\\
z_j &= x + 2h\mathsf{e}_j, & z_{n+j} &= x - 2h\mathsf{e}_j, & j &= 1,\dots,n,\\
y_{ij} &= x + h(\mathsf{e}_i + \mathsf{e}_j), & y_{ji} &= x - h(\mathsf{e}_i + \mathsf{e}_j), & 1 &\le i < j \le n,\\
w_{ij} &= x - h(\mathsf{e}_i - \mathsf{e}_j), & w_{ji} &= x + h(\mathsf{e}_i - \mathsf{e}_j), & 1 &\le i < j \le n.
\end{aligned}$$



$$L_h(u)(x) = q\,u(x) + \sum_{j=1}^{2n} \alpha_j \left(u(x) - u(x_j)\right) + \sum_{j=1}^{2n} \beta_j \left(u(x) - u(z_j)\right) + \sum_{\substack{i,j=1\\ i\ne j}}^{n} \gamma_{ij} \left(u(x) - u(y_{ij})\right) + \sum_{\substack{i,j=1\\ i\ne j}}^{n} \delta_{ij} \left(u(x) - u(w_{ij})\right)$$

▷ L_h is quasi-symmetric if $\gamma_{ij} = \gamma_{ji}$, $\delta_{ij} = \delta_{ji}$, $\beta_j = \beta_{n+j}$

▷ L_h is symmetric if it is quasi-symmetric and $\alpha_j = \alpha_{n+j}$

▷ L_h is of non negative type if $q, \alpha_j, \beta_j, \gamma_{ij}, \delta_{ij} \ge 0$

▷ L_h is of positive type if it is of non negative type and $h^2\alpha_j \ge C$

Second Order Difference Operators

- ▷ Homogeneous Fields on Γ_h: $\mathsf{b}(x) = \sum_{j=1}^{2n} b_j\, e_{xx_j}$ and $\mathsf{A}(x) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$
- ▷ First order difference operators: $\nabla u = \frac{1}{h}\,\mathsf{d}u$ and $\mathsf{div}\,\mathsf{f}(x) = \frac{1}{h}\sum_{j=1}^{2n} f^a(x, x_j)$
- ▷ Second order difference operator: $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$
- ⋆ b and A can be chosen s.t. $b_{n+j} = -b_j$ and $A_2 = A_2^t$, $A_3 = A_3^t$ and $A_4 = A_1^t$

$$
\begin{aligned}
\mathcal{L}_h(u)(x) = {} & \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}(a_{ij}^1 + a_{ij}^3) - hb_j\Big)\Big(u(x) - u(x_j)\Big) \\
& + \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}(a_{ji}^1 + a_{ji}^2) + hb_j\Big)\Big(u(x) - u(x_{n+j})\Big) \\
& - \frac{1}{2h^2}\sum_{j=1}^{n} a_{jj}^3\Big(u(x) - u(z_j)\Big) - \frac{1}{2h^2}\sum_{j=1}^{n} a_{jj}^2\Big(u(x) - u(z_{n+j})\Big) \\
& - \frac{1}{h^2}\sum_{1\le i<j\le n} a_{ij}^3\Big(u(x) - u(y_{ij})\Big) - \frac{1}{h^2}\sum_{1\le i<j\le n} a_{ji}^2\Big(u(x) - u(y_{ji})\Big) \\
& - \frac{1}{h^2}\sum_{\substack{i,j=1 \\ i\ne j}}^{n} a_{ij}^1\Big(u(x) - u(w_{ij})\Big) + qu(x), \qquad x \in V_h.
\end{aligned}
$$

Equivalence Scheme-Difference Operators

▷ If L_h is a difference scheme with constant coefficients

$$L_h(u)(x) = q\,u(x) + \sum_{j=1}^{2n} \alpha_j \big(u(x) - u(x_j)\big) + \sum_{j=1}^{2n} \beta_j \big(u(x) - u(z_j)\big) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \gamma_{ij} \big(u(x) - u(y_{ij})\big) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \delta_{ij} \big(u(x) - u(w_{ij})\big)$$

then, there exist a unique homogeneous field of matrices A and a unique homogeneous flow b such that $L_h(u) = -\mathrm{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + q\,u$.

▷ Relation between coefficients of the schemes and the fields

$$\left.\begin{aligned}
a^1_{ij} = -h^2\delta_{ij},\ \ a^1_{ji} = -h^2\delta_{ji},\ \ a^2_{ji} = -h^2\gamma_{ji},\ \ a^3_{ij} = -h^2\gamma_{ij},\\
a^2_{jj} = -2h^2\beta_{n+j},\ \ a^3_{jj} = -2h^2\beta_j,\\
a^1_{jj} - hb_j = h^2\Big(\alpha_j + 2\beta_j + \sum_{i=1}^{j-1}(\gamma_{ij}+\delta_{ij}) + \sum_{i=j+1}^{n}(\gamma_{ji}+\delta_{ij})\Big),\\
a^1_{jj} + hb_j = h^2\Big(\alpha_{n+j} + 2\beta_{n+j} + \sum_{i=1}^{j-1}(\gamma_{ji}+\delta_{ji}) + \sum_{i=j+1}^{n}(\gamma_{ij}+\delta_{ji})\Big)
\end{aligned}\right\}$$

Symmetry and positivity

▷ If $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u \rangle + qu$, the following properties hold:

i) $\mathcal{L}_h$ is a quasi-symmetric scheme iff A is symmetric. Moreover, $\mathcal{L}_h$ is a symmetric scheme iff A is symmetric and $\mathsf{b} = 0$

ii) $\mathcal{L}_h$ is a non negative scheme iff $q \geq 0$, A is a Z-field and $\mathsf{r}_\mathsf{A} \geq -h\mathsf{b}$ for h small enough. In particular, when $\mathcal{L}_h$ is a non negative scheme it must be verified that $\mathsf{d}_\mathsf{A} \geq h|\mathsf{b}|$ for h small enough.

iii) If $\lim_{h\to 0} h\mathsf{b} = 0$, then $\mathcal{L}_h$ is of positive type iff $q \geq 0$ and A is an s.d.d. M-field

iv) If $\mathsf{b} = 0$, then $\mathcal{L}_h$ is a scheme of non negative type iff $q \geq 0$ and A is a d.d. M-field

v) If $\mathcal{L}_h$ is a quasi-symmetric scheme then it is of non negative type iff $q \geq 0$ and A is a symmetric d.d. M-field with $\mathsf{r}_\mathsf{A} \geq h|\mathsf{b}|$

Consistency

- ▷ Differential operator: $\mathcal{L}u = -\mathsf{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u$
- ▷ Differential operator: $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$
- ▷ Consistency: For all $x \in V$:

$$L_h(u)(x) - L(u)(x) = \phi^0\, u(x) + \sum_{k=1}^{m} \frac{h^k}{k!}\left[\sum_{j=1}^{n} \phi_j^k\, D_j^k u(x) + \sum_{l=1}^{k-1} \binom{k}{l} \sum_{1\le i<j\le n} \psi_{ij}^{lk-l}\, D_i^l D_j^{k-l} u(x)\right] + T_{m+1}(x),$$

▷ Conditions to obtain consistency:

$m = 1$ $\quad \phi^0 = q - k_0, \quad h\phi_j^1 = -(2b_j + k_j)$

$m = 2$ $\quad h^2\phi_j^2 = 2a_{jj}^1 - a_{jj}^2 - a_{jj}^3 - 2k_{jj}, \quad h^2\psi_{ij}^{11} = a_{ij}^1 + a_{ji}^1 - a_{ij}^2 - a_{ij}^3 - 2k_{ij}$

$m = 3$ $\quad h^2\phi_j^3 = -2hb_j + 3(a_{jj}^2 - a_{jj}^3)$

$$h^2\psi_{ij}^{12} = a_{ij}^1 - a_{ji}^1 + a_{ij}^2 - a_{ij}^3, \quad h^2\psi_{ij}^{21} = a_{ji}^1 - a_{ij}^1 + a_{ij}^2 - a_{ji}^3$$

Second order difference operators

▷ $L(u) = -\mathsf{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u,\ K \neq 0$

▷ $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$ is a consistent scheme iff

$\exists\ \ M \in O(h^s),\ s \in (-2, \min\{0, r-1\}],\ \phi^0, \phi^1, \Phi, h\Psi \in O(h^r),\ r > 0,$ where M, Φ are symmetric such that $q = k_0 + \phi^0$, $\mathsf{b} = -\frac{1}{2}\,(\hat{\mathsf{k}} + \hat{\phi}^1)$,

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} - \begin{bmatrix} D & 2D \\ 0 & D \end{bmatrix} + \begin{bmatrix} \Phi + \Psi & \Psi + \Psi^t \\ 0 & \Phi + \Psi^t \end{bmatrix}$$

D is the diagonal $d_{jj} = \frac{h}{6}\,(k_j + h\phi^1_j)$. Consistency is $\min\{r, s+2\} \leq 2$

▷ Symmetry $\Longrightarrow\ \ \Psi = D$

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}$$

$s \in (-2, 0]$. Then, $\mathcal{L}_h$ is a quasi-symmetric consistent scheme of order $\min\{r, s+2\}$ and when $\mathsf{k} \neq 0$, 2 is the greatest order of consistency. Moreover, L has symmetric consistent schemes iff it is selfadjoint

Symmetry, Positivity and Consistency

▷ L has not consistent schemes of non negative type of order greater than 2

▷ If L has consistent schemes of non negative type, then $k_0 \geq 0$ and K is positive semidefinite and diagonalment dominant, that is

$$k_0 \geq 0 \quad \text{and} \quad k_{jj} \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |k_{ji}|, \qquad j = 1, \ldots, n$$

Moreover, the converse holds when L is self-adjoint

▷ L has consistent schemes of positive type iff $k_0 \geq 0$ and K is positive definite and strictly diagonalment dominant, that is iff

$$k_0 \geq 0 \quad \text{and} \quad k_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n} |k_{ji}|, \qquad j = 1, \ldots, n$$

Symmetry, Positivity and Consistency

▷ If K is a positive definite s.d.d matrix and

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} - \begin{bmatrix} D & 2D \\ 0 & D \end{bmatrix} + \begin{bmatrix} \Phi + \Psi & \Psi + \Psi^t \\ 0 & \Phi + \Psi^t \end{bmatrix}$$

Then $\mathcal{L}_h$ is a $\min\{r, 2\}$-consistent scheme of positive type if

i) If $\lim_{h \to 0} M(h) < -K^+$ and $\lim_{h \to 0} \sum_{i=1}^{n} m_{ji}(h) > -\frac{1}{2} \sum_{i=1}^{n} k_{ji}$, $j = 1, \ldots, n$.

ii) If $M \leq -K^+$, $\Phi \leq 0$, $\Psi \leq D$ and $\lim_{h \to 0} \sum_{i=1}^{n} m_{ji}(h) > -\frac{1}{2} \sum_{i=1}^{n} k_{ji}$,
$j = 1, \ldots, n$.

▷ The differential operator L admits consistent difference schemes $\mathcal{L}_h$ such that A is a metric tensor iff L is a semi-elliptic operator, that is, K is a positive semi-definite matrix

Some widely used examples

▷ $\mathcal{L}_h(u) = -\mathrm{div}\,(\mathsf{A}\nabla u) + -\frac{1}{2}\langle \hat{\mathsf{k}}, \nabla u\rangle + k_0 u$

$$A = \begin{bmatrix} K+M & M \\ M & K+M \end{bmatrix}$$

- $\mathcal{L}_h$ is a 2-consistent quasi-symmetric scheme and when $\mathsf{k} \neq 0$, 2 is the greatest order of consistency for this type of schemes

- $$\mathcal{L}_h(u) = \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}(k_{ij} + 2m_{ij})\Big)\Big(2u(x) - u(x_j) - u(x_{n+j})\Big)$$

$$-\frac{1}{h^2}\sum_{1\le i<j\le n}^{n}\Big((k_{ij} + m_{ij})\Big(2u(x) - u(w_{ij}) - u(w_{ji})\Big) + m_{ij}\Big(2u(x) - u(y_{ij}) - u(y_{ji})\Big)\Big)$$

$$\frac{1}{2h^2}\sum_{j=1}^{n} m_{jj}\Big(2u(x) - u(z_j) - u(z_{n+j})\Big) - \frac{1}{2h}\sum_{j=1}^{n} k_j\Big(u(x_j) - u(x_{n+j})\Big) + k_0 u(x).$$

Some widely used examples

▷ Standard difference scheme: $M = 0$

$$\mathcal{L}_h(u) = \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} k_{ij}\Big)\Big(2u(x) - u(x_j) - u(x_{n+j})\Big)$$

$$-\frac{1}{h^2}\sum_{1\le i<j\le n}^{n} k_{ij}\Big(2u(x) - u(w_{ij}) - u(w_{ji})\Big) - \frac{1}{2h}\sum_{j=1}^{n} k_j\Big(u(x_j) - u(x_{n+j})\Big) + k_0 u(x)$$

- Positive type iff $k_0 \geq 0$ and K is a symmetric and s.d.d. M-matrix

▷ $m_{jj} = 0$ and $m_{ij} = -k_{ij}$

$$\mathcal{L}_h(u) = \frac{1}{h^2}\sum_{j=1}^{n}\Big(k_{jj} - \sum_{\substack{i=1\\ i\neq j}}^{n} k_{ij}\Big)\Big(2u(x) - u(x_j) - u(x_{n+j})\Big)$$

$$+\frac{1}{h^2}\sum_{1\le i<j\le n}^{n} k_{ij}\Big(2u(x) - u(y_{ij}) - u(y_{ji})\Big) - \frac{1}{2h}\sum_{j=1}^{n} k_j\Big(u(x_j) - u(x_{n+j})\Big) + k_0 u(x)$$

- Positive type iff $k_0 \geq 0$ and K is a non negative and s.d.d. matrix

Some widely used examples

▷ Cross scheme: $m_{jj} = 0$, $m_{ij} = -\frac{1}{2(n-1)}\Big(k_{jj} + (n-1)\,k_{ij}\Big)$

$$\begin{aligned}\mathcal{L}_h(u) = {} & \frac{1}{2(n-1)\,h^2} \sum_{1\le i<j\le n}^{n} \Big(k_{jj} - (n-1)\,k_{ij}\Big)\Big(2u(x) - u(w_{ij}) - u(w_{ji})\Big) \\ & + \frac{1}{2(n-1)\,h^2} \sum_{1\le i<j\le n}^{n} \Big(k_{jj} + (n-1)\,k_{ij}\Big)\Big(2u(x) - u(y_{ij}) - u(y_{ji})\Big) \\ & - \frac{1}{2h}\sum_{j=1}^{n} k_j\Big(u(x_j) - u(x_{n+j})\Big) + k_0 u(x).\end{aligned}$$

- Positive type iff $k_0 \ge 0$ and $k_{jj} \ge (n-1)\,|k_{ij}|$, $1 \le i < j \le n$, which in particular implies that K is an s.d.d. matrix

Some widely used examples: K diagonal

▷ Generalization of the nine-point scheme: $m_{jj} = 0,\ m_{ij} = -\dfrac{k_{ii} + k_{jj}}{12(n-1)}$

$$
\begin{aligned}
\mathcal{L}_h(u) = {} & \frac{1}{6(n-1)h^2} \sum_{j=1}^{n} \Big(5(n-1)\, k_{jj} - \sum_{\substack{i=1 \\ i \neq j}}^{n} k_{ii} \Big) \Big(2u(x) - u(x_j) - u(x_{n+j}) \Big) \\
& + \frac{1}{12(n-1)h^2} \sum_{1 \le i < j \le n}^{n} (k_{ii} + k_{jj}) \Big(4u(x) - u(w_{ij}) - u(w_{ji}) - u(y_{ij}) - u(y_{ji}) \Big) \\
& - \frac{1}{2h} \sum_{j=1}^{n} k_j \Big(u(x_j) - u(x_{n+j}) \Big) + k_0 u(x)
\end{aligned}
$$

- If $k_{ss} = \min\limits_{j=1,\ldots,n} \{k_{jj}\}$, then $\mathcal{L}_h(u)$ is of positive type iff $k_0 \ge 0$, $k_{jj} \ge 0$ and

$$5k_{ss} > \frac{1}{n-1} \sum_{\substack{i=1 \\ i \neq s}}^{n} k_{ii}$$

Some widely used examples: K diagonal and $\mathsf{k} = 0$

▷ 4-consistent scheme: $M = \frac{1}{6} K$

$$\mathcal{L}_h(u) = \frac{4}{3h^2} \sum_{j=1}^{n} k_{jj} \Big(2u(x) - u(x_j) - u(x_{n+j}) \Big) - \frac{1}{12h^2} \sum_{j=1}^{n} k_{jj} \Big(2u(x) - u(z_j) - u(z_{n+j}) \Big) - k_0 u(x)$$

- It is not of non negative type

Cohomology of a discrete Riemannian manifold

▷ De Rham complex: $\nabla_0 = \nabla, \quad \nabla_{2j+1}\mathsf{f} = 2\mathsf{A}(\mathsf{Mf})^s, \quad \nabla_{2j}\mathsf{f} = -2\mathsf{A}(\mathsf{Mf})^a$

$$\{0\} \xrightarrow{0} \mathcal{C}(V) \xrightarrow{\nabla_0} \mathcal{X}(\Gamma) \xrightarrow{\nabla_1} \mathcal{X}(\Gamma) \cdots \xrightarrow{\nabla_n} \mathcal{X}(\Gamma) \xrightarrow{\nabla_{n+1}} \cdots, \qquad \nabla_n \circ \nabla_{n-1} = 0$$

▷ n-th de Rham cohomology group: $H^n(\Gamma) = \ker\nabla_n / \mathrm{Img}\nabla_{n-1}$

▷ n-th Betti number: $\beta_n = \dim H^n(\Gamma)$

⋆ $\beta_0 = m, \quad \beta_1 = |E| - |V| + m, \quad \beta_n = 0, \quad n \geq 2 \quad \Longrightarrow \quad \chi(\Gamma) = \sum_{n=0}^{\infty} (-1)^n \beta_n$

▷ $\delta_n = \nabla_n^* \quad \Longrightarrow \quad \delta_0 = -\mathsf{div}\,, \quad \delta_{2j+1}\mathsf{f} = \frac{2}{\mu}(\mu\mathsf{f})^s, \quad \delta_{2j} = -\frac{2}{\mu}(\mu\mathsf{f})^a$

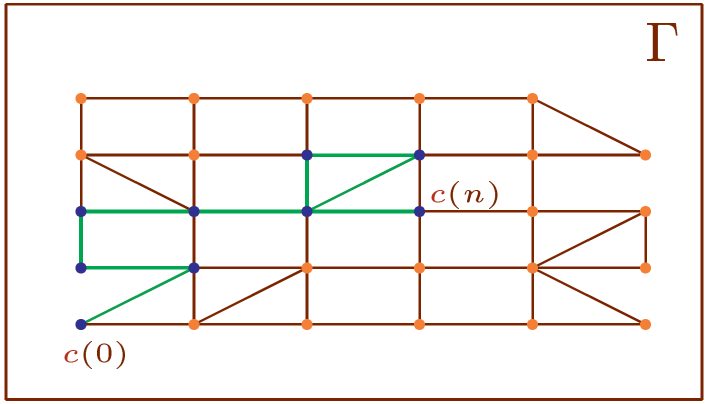
$$\{0\} \xleftarrow{0} \mathcal{C}(V) \xleftarrow{\delta_0} \mathcal{X}(\Gamma) \xleftarrow{\delta_1} \mathcal{X}(\Gamma) \cdots \xleftarrow{\delta_n} \mathcal{X}(\Gamma) \xleftarrow{\delta_{n+1}} \cdots, \qquad \delta_{n-1} \circ \delta_n = 0$$

▷ Hodge's Laplacian: $\Delta_n = \delta_n \circ \nabla_n + \nabla_{n-1} \circ \delta_{n-1} \quad \Longrightarrow \quad \Delta_0 = -\Delta$

⋆ Hodge's Decomposition: $\mathcal{X}(\Gamma) = \ker\Delta_n \oplus \mathrm{Img}\nabla_{n-1} \oplus \mathrm{Img}\delta_n$

$$H^n(\Gamma) \simeq \ker\Delta_n \quad \Longrightarrow \quad H^1(\Gamma) \simeq \{\mathsf{f} \in \mathcal{X}(\Gamma) : \mathsf{div}\,\mathsf{f} = 0, \ \mathsf{curl}\,\mathsf{f} = 0\}$$

Integration along curves



▷ Curve: $\gamma(j) = (c(j), \dot{c}(j)), \quad j = 0, \dots, n$

- $\dot{c}(n) = 0 \in T_{c(n)}(\Gamma)$
- $\dot{c}(j) = \alpha_j e_{c(j)c(j+1)}, \; j = 0, \dots, n-1$
- $\exists\, r(\gamma) \in \mathbb{R}$, s.t. $\alpha_j \mu(c(j)) = 2r(\gamma), \; j = 0, \dots, n-1$

⋆ Tangent field to γ : $\dot{\gamma} = \sum_{j=0}^{n} \dot{c}(j) \Longrightarrow \operatorname{div} \dot{\gamma} = \frac{r(\gamma)}{\nu}\left(\varepsilon_{c(0)} - \varepsilon_{c(n)}\right)$

▷ Circulation of $\mathsf{f} \in \mathcal{X}(\Gamma)$ along γ: $\int_\gamma \mathsf{f} = \frac{1}{2}\int_V \langle (\mathsf{Mf})^a, \dot{\gamma}\rangle \, d\mu$

⋆ Stokes-Ampère Theorem:

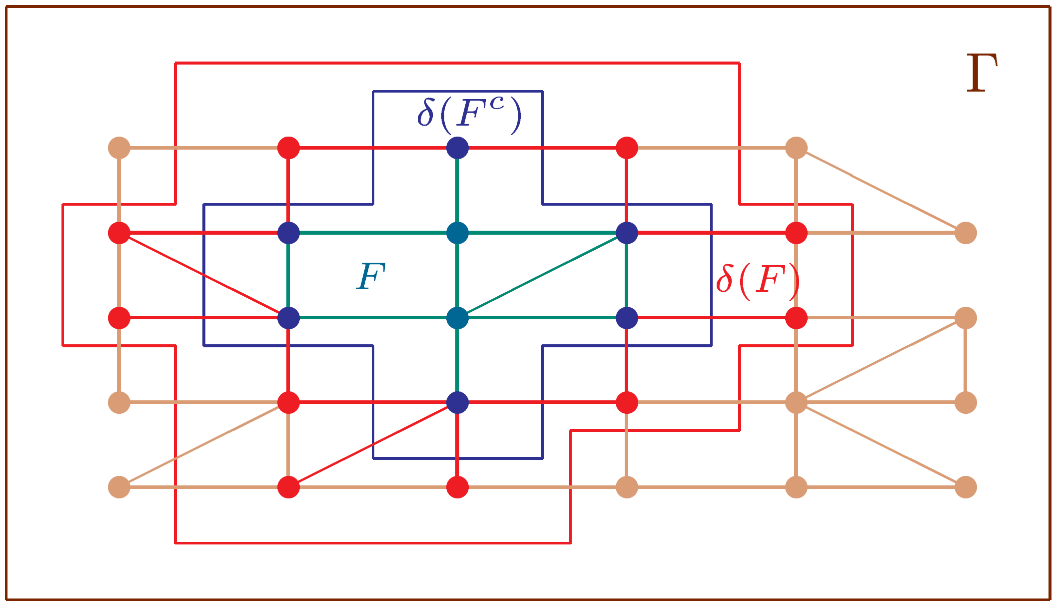
$$\mathrm{sop}(\mathsf{f}) \cap \mathrm{sop}(\dot{\gamma}) = \emptyset \Longrightarrow \int_\gamma \mathsf{f} = \int_{\mathrm{sop}(\mathsf{f})} \langle \operatorname{curl} \mathsf{f}, \mathsf{n}_\gamma \rangle \, d\mu, \qquad \mathsf{n}_\gamma = -\frac{1}{4}(\mu\, \dot{\gamma})^s$$

⋆ $\mathsf{f} \in \mathcal{X}(\Gamma)$ is a conservative field iff there exists $u \in \mathcal{C}(V)$ such that $(\mathsf{Mf})^a = \mathrm{d}u$

$$\Downarrow$$

If $\operatorname{curl} \mathsf{f} = 0$, then f is a conservative field iff there exists $u \in \mathcal{C}(V)$ s. t. $\mathsf{f} = \nabla u$

Divergence Theorem

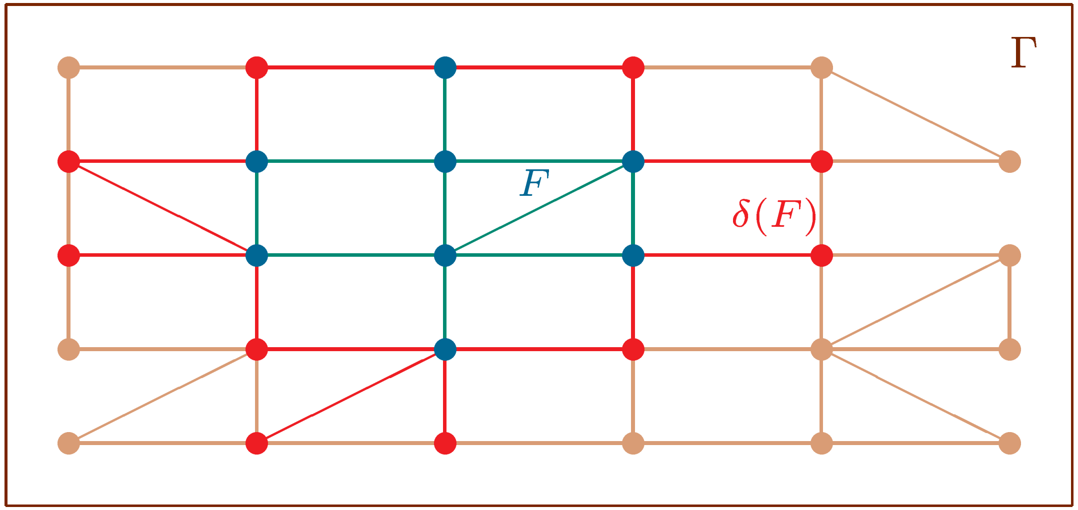


▷ Normal vector field to F: $\mathsf{n}_F = -\mathsf{d}\chi_F \implies \mathsf{sop}(\mathsf{n}_F) = \delta(F) \cup \delta(F^c)$

$$\int_F \mathsf{div}\,\mathsf{f}\,d\nu = \int_{\delta(F^c)} \langle \mathsf{n}_F, (\mu\mathsf{f})^a \rangle = \int_{\delta(F)} \langle \mathsf{n}_F, (\mu\mathsf{f})^a \rangle$$

⋆ Si $\mu\mathsf{f} \in \mathcal{X}^a(\Gamma) \implies \int_F \mathsf{div}\,\mathsf{f}\,d\nu = \int_{\delta(F^c)} \langle \mathsf{n}_F, \mathsf{f} \rangle\, d\mu = \int_{\delta(F)} \langle \mathsf{n}_F, \mathsf{f} \rangle\, d\mu$

Green's Identities



▷ $q_F(x) = \sum_{y \in \mathsf{Ext}(F)} c(x,y), \quad x \in F$

▷ $b = c \cdot \chi_{(F\times\bar{F})\cup(\bar{F}\times F)}$

⋆ A(x) Stieltjes d.d., $x \in \bar{F} \Rightarrow b \geq 0$

▷ Conormal derivative of $u \in \mathcal{C}(\bar{F})$: $\dfrac{\partial u}{\partial \eta_F}(x) = \dfrac{1}{\nu(x)} \sum_{y\in F} c(x,y)\big(u(x) - u(y)\big), \ \ x \in \delta(F)$

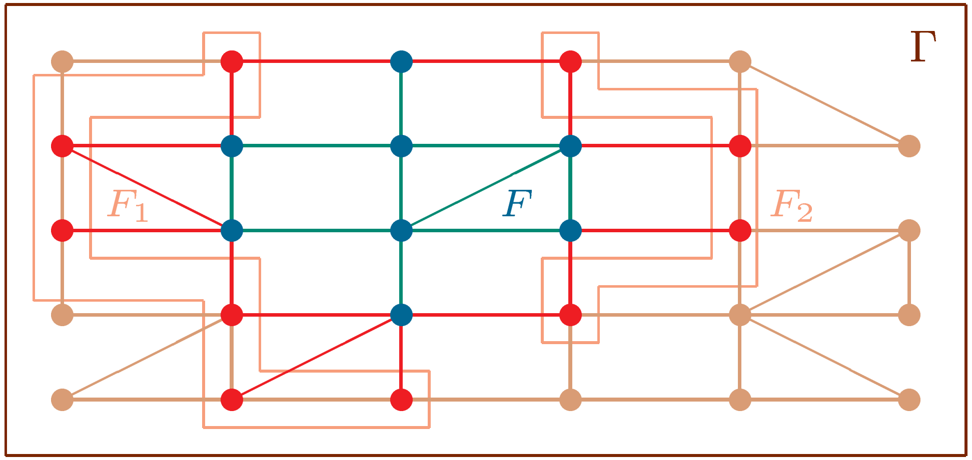
⋆ $u \in \mathcal{C}(\bar{F}) \implies \Delta u(x) = \dfrac{1}{\nu(x)} \sum_{y \in \bar{F}} c(x,y)\big(u(y) - u(x)\big) - \dfrac{1}{\nu(x)} q_F(x)\, u(x), \quad x \in F$

⋆ Green's Identities: For each $u, v \in \mathcal{C}(\bar{F})$

$$-\int_F v\Delta u \, d\nu = \frac{1}{2}\int_{\bar{F}\times\bar{F}} b(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, dx\, dy + \int_F q_F\, u\, v - \int_{\delta(F)} v \frac{\partial u}{\partial \eta_F}\, d\nu$$

$$\int_F (v\Delta u - u\Delta v)\, d\nu = \int_{\delta(F)} \left(v\frac{\partial u}{\partial \eta_F} - u \frac{\partial v}{\partial \eta_F}\right) d\nu$$

Boundary Value Problems



▷ $\delta_c(F) = \left\{ y \in \delta(F) : y \sim x \text{ con } x \in F \text{ y } c(x,y) \neq 0 \right\}$

▷ $\delta_c(F) = F_1 \cup F_2, \quad F_1 \cap F_2 = \emptyset, \qquad q \in \mathcal{C}(F), \quad h \in \mathcal{C}(F_1)$

▷ Mixed Dirichlet-Robin Problem: $f \in \mathcal{C}(F), \quad g_1 \in \mathcal{C}(F_1), \quad g_2 \in \mathcal{C}(F_2)$

$$\left.\begin{aligned} -\Delta u(x) + \frac{1}{\nu(x)}\, q(x)u(x) &= f(x), \quad x \in F \\ \frac{\partial u}{\partial \eta_F}(x) + \frac{1}{\nu(x)}\, h(x)u(x) &= g_1(x), \quad x \in F_1 \\ u(x) &= g_2(x), \quad x \in F_2 \end{aligned}\right\} P$$

⋆ P is self-adjoint

Existence and uniqueness of solution

⋆ P has solution iff $\displaystyle\int_F f v\, d\nu + \int_{F_1} g_1 v\, d\nu = \int_{F_2} g_2 \frac{\partial v}{\partial \eta_F}\, d\nu$ for each $v \in \mathcal{V}_H$

and has a unique solution u such that $u \in \mathcal{V}_H^{\perp}$

▷ $\sigma \in \mathcal{C}(\bar{F})$ such that $\sigma(x) > 0, \quad x \in \bar{F}$

▷ $\displaystyle q_\sigma(x) = \frac{\nu(x)}{\sigma(x)} \Delta\sigma(x), \quad x \in F, \qquad h_\sigma(x) = -\frac{\nu(x)}{\sigma(x)} \frac{\partial \sigma}{\partial \eta_F}(x), \quad x \in F_1$

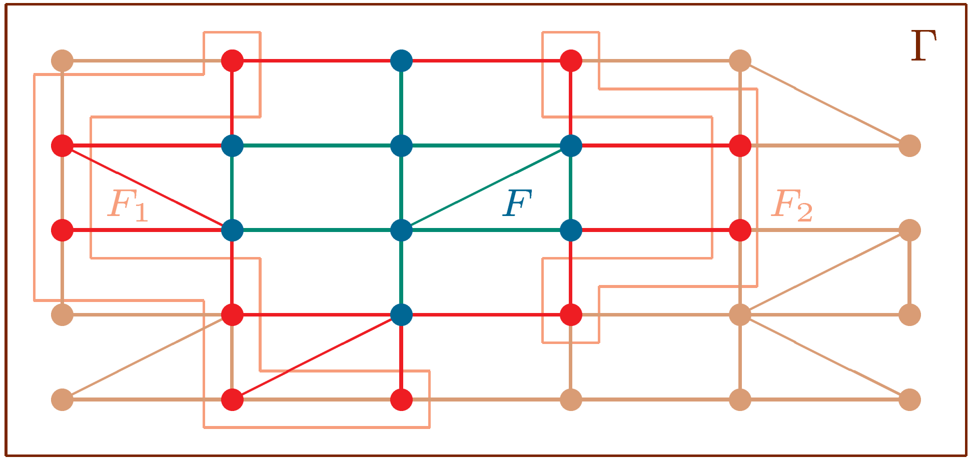
▷ Hypothesis : $q \geq q_\sigma$ and $h \geq h_\sigma$

- Si $F_2 = \emptyset$, $q = q_\sigma$ and $h = h_\sigma$, P has solution iff $\displaystyle\int_F f\sigma\, d\nu + \int_{F_1} g_1 \sigma\, d\nu = 0$
- Otherwise, P has a unique solution

⋆ Dirichlet Principle: u is a solution of P iff it minimize $\mathcal{J}$ on $\left\{ v \in \mathcal{C}(\bar{F}) : v_{|_{F_2}} = g_2 \right\}$

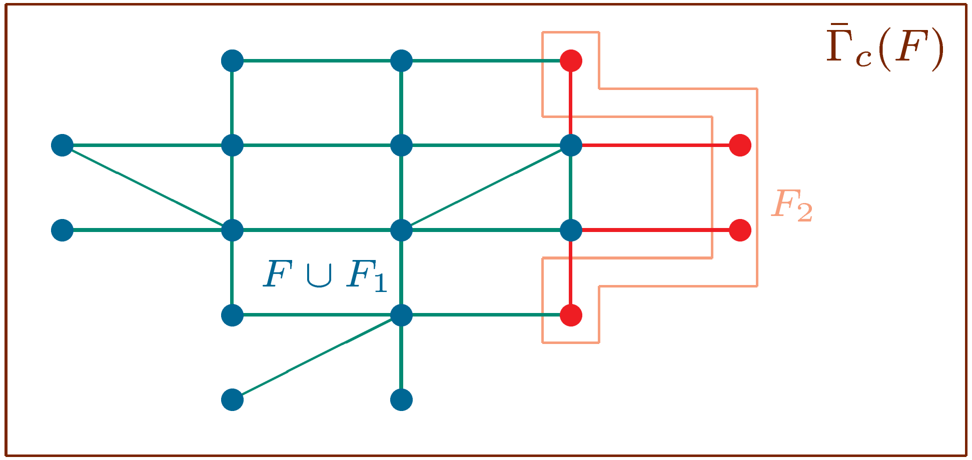
$$\mathcal{J}(v) = \frac{1}{2} \int_{\bar{F}\times\bar{F}} b(x,y)\big(v(y) - v(x)\big)^2 dx\, dy + \int_F (q_F + q) v^2 + \int_{F_1} h v^2 - 2\int_F f v\, d\nu - 2 \int_{F_1} g_1 v\, d\nu$$

Equivalent Boundary value Problem



$$P\begin{cases} -\Delta u(x) + \dfrac{1}{\nu(x)}\, q(x)u(x) = f(x), & x \in F \\ \dfrac{\partial u}{\partial \eta_F}(x) + \dfrac{1}{\nu(x)}\, h(x)u(x) = g_1(x), & x \in F_1 \\ u(x) = g_2(x), & x \in F_2 \end{cases}$$

$$\Updownarrow$$



$$\bar{P}\begin{cases} -\bar{\Delta} u(x) + \dfrac{1}{\nu(x)}\, \bar{q}(x)u(x) = \bar{f}(x), & x \in F \cup F_1 \\ u(x) = g_2(x), & x \in F_2 \end{cases}$$

$$\bar{q} = \begin{cases} q + q_F & \text{in } F \\ h & \text{in } F_1 \end{cases} \qquad \bar{f} = \begin{cases} f & \text{in } F \\ g_1 & \text{in } F_1 \end{cases} \qquad \bar{\Delta} = \begin{cases} \Delta + \dfrac{1}{\nu} q_F & \text{in } F \\ \dfrac{\partial}{\partial \eta_F} & \text{in } F_1 \end{cases}$$

Dirichlet Problem and Poisson equation

▷ $q \in \mathcal{C}(V)$, $\mathcal{L} = -\Delta + q$, $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(F^c)$ $\Longrightarrow$ $\left.\begin{aligned} \mathcal{L}(u)(x) &= f, \quad x \in F \\ u(x) &= g, \quad x \in F^c \end{aligned}\right\} P$

▷ V is connected in $\Gamma_c(V)$, $c \geq 0$ and $\exists\, \sigma \in \mathcal{C}(V)$ s.t. $\sigma(x) > 0$, $x \in V$ and $q \geq q_\sigma$

▷ $\emptyset \neq F \subset V$ and it is not simultaneously verified that $F = V$ and $q = q_\sigma$

⋆ Minimum Principle: $\mathcal{L}(u)_{|_F} \geq 0$ y $u_{|_{F^c}} \geq 0 \Longrightarrow u \geq 0$

⋆ $f \in \mathcal{C}^+(F) \Longrightarrow u \in \mathcal{C}^+(F)$ y $\mathsf{sop}(f) \subset \mathsf{sop}(u)$